2-level longitudinal models with random time origin and complex level 1 variance structure

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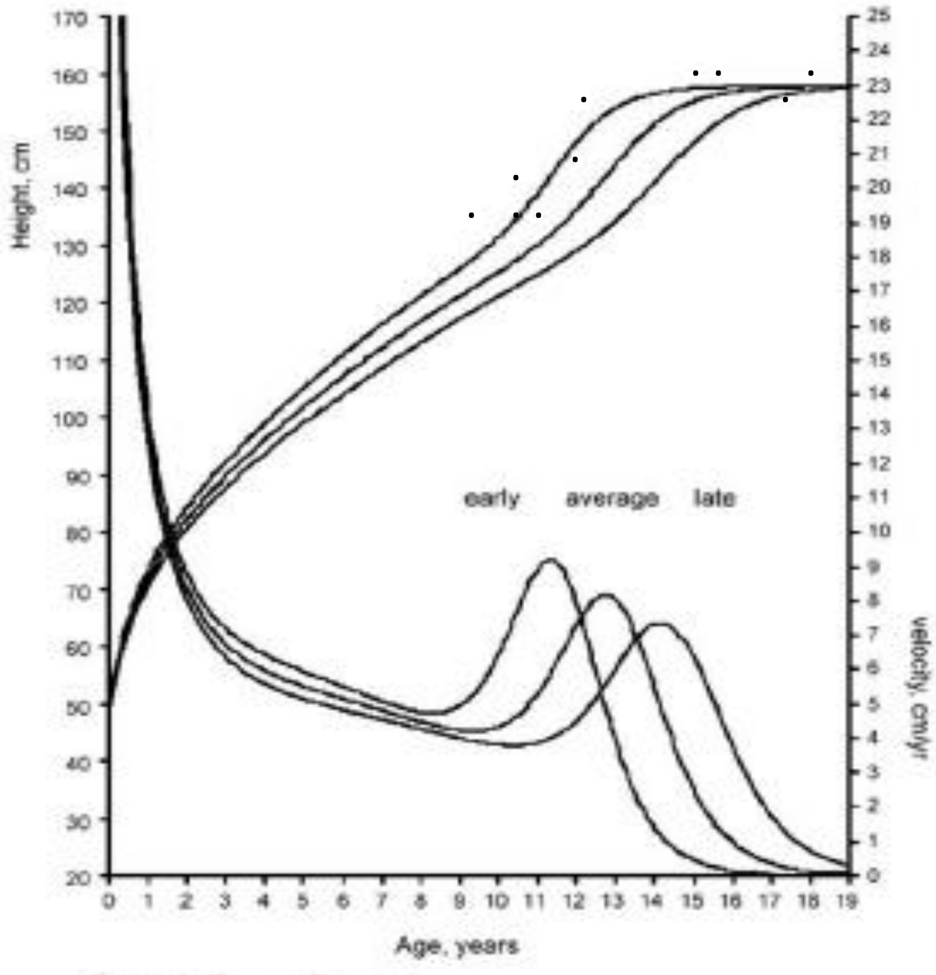
Childrens’ growth curves



Note:

a). curves similar shape with displaced origin (tempo) and scale

b). Level 1 variation (about growth trend) tends to zero at maturity

An individual 3-random-effects model (SITAR model – Cole et al.)

- $y_{ij} = \beta_{0j} + f(x_{ij}^*) + e_{ij}$ Overall size
- $x_{ij}^* = e^{\gamma_j}(x_{ij} - v_j)$ Shape (velocity) (1)
- $\beta_{0j} = \beta_0 + u_{0j}$ Age origin
- $f()$ is a suitable growth function of transformed age (x_{ij}^*)
- Cole et al. use a natural cubic spline – we use the regression spline – an underlying polynomial with additional smoothly joining cubics.
- $\begin{pmatrix} \beta_{0j} \\ \gamma_j \\ v_j \end{pmatrix} \sim MVN\left(\begin{pmatrix} \beta_0 \\ \gamma_0 \\ v_0 \end{pmatrix}, \Omega_u\right), \quad \Omega_u = \begin{pmatrix} \sigma_\beta^2 & & \\ \sigma_{\beta\gamma} & \sigma_\gamma^2 & \\ \sigma_{\beta v} & \sigma_{\gamma v} & \sigma_v^2 \end{pmatrix}, \quad e_{ij} \sim N(0, \sigma_e^2)$

Cole, TJ., Pan, H. and Butler, GE. (2014). *Ann. Hum. Biol.*, 41(1), 76-83 (max likelihood estimation in R)

A generalisation

- Assuming for simplicity a basic cubic we can rewrite (1) as
- $y_{ij} = \beta_{0j} + e^{\gamma_j}(x_{ij} - v_j) + e^{2\gamma_j}(x_{ij} - v_j)^2 + e^{3\gamma_j}(x_{ij} - v_j)^3 + regn\ spline\ terms + e_{ij}$
- More flexibly, unconstraining coefficients, we have
- $y_{ij} = \beta_{0j} + \beta_{1j}(x_{ij} - v_j) + \beta_{2j}(x_{ij} - v_j)^2 + \beta_{3j}(x_{ij} - v_j)^3 + regn\ spline\ terms + e_{ij}$ (2)
- So that we now have 5 (correlated) individual level random effects.
- We will also allow the level 1 residual variance to depend on covariates with 1 or more random effects (See Leckie et al., JRSSA forthcoming, for more details)
- $e_{ij} \sim N(0, \sigma^2_{eij})$, $\log(\sigma^2_{eij}) = \gamma_0 + \gamma_1 z_{ij} + \gamma_2 z^2_{ij} + w_j$
- $z_{ij} = x_{ij} - v_j$
- This adds a further random effect (w_j) at level 2. Importantly it allows the level 1 variance to tend to residual measurement error at maturity.

Level 2 specification

- $$\begin{pmatrix} \beta_{0j} \\ \beta_{1j} \\ \beta_{2j} \\ \beta_{3j} \\ v_j \\ w_j \end{pmatrix} \sim MVN\left(\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ v_0 \\ w_0 \end{pmatrix}, \Omega_{uvw}\right)$$

- $$\Omega_{uvw} = \begin{pmatrix} \sigma_{u0}^2 & & & & & \\ \sigma_{u01} & \sigma_{u1}^2 & & & & \\ \sigma_{u02} & \sigma_{u12} & \sigma_{u2}^2 & & & \\ \sigma_{u03} & \sigma_{u13} & \sigma_{u23} & \sigma_{u3}^2 & & \\ \sigma_{u0v} & \sigma_{u1v} & \sigma_{u2v} & \sigma_{u3v} & \sigma_v^2 & \\ \sigma_{u0w} & \sigma_{u1w} & \sigma_{u2w} & \sigma_{u3w} & \sigma_{vw} & \sigma_w^2 \end{pmatrix}$$

Estimation

- MCMC with diffuse priors.
- Gibbs steps update the fixed coefficients, covariance matrix and the level 2 random effects in the polynomial function
- MH steps to update the other random effects and the coefficients of the variance function.
- Good starting values (for polynomial fixed effects) are important. Several chains with different starting values used.
- Implemented experimentally in STATJR (CMM Bristol).

Results: 101 boys aged 8-18, height, Approx. 6 months apart.

5th degree polynomial (not all shown and spline terms omitted) – coefficients up to cubic random. Burnin=iterations=25,000. Age centered on 12.7 years. Term in brackets for non-age-shifted model

β_0	163.1	0.64
β_1	10.3 (7.6)	0.09
β_2	2.71	0.08
β_3	0.26	0.007
Age shift	1.37	0.18
Variance function intercept	-1.29	0.08
Variance function slope	-0.051	0.025
Variance function quadratic	-0.023	0.006
DIC (PD)	2849.7	478.3

- For example, if we have a boy who has an average downward shift equal to the mean shift of 1.37, at an actual age of 9.0 years (2.29 years earlier than his age origin), the expected variance is 0.24. For such a boy at actual age of 16.0 years, however, the expected variance is 0.07.
- For earlier ages we have similar results for when the variance function is defined on actual age, with the expected variance at age 9.0 years of 0.30. For later ages, however, the variance does not decrease as before: at age 16, for example, it is estimated as 0.22 and this reflects the variability in the timing of the growth spurt.

Random effects

Ω_u	$\begin{pmatrix} 53.2 & & & & & \\ 0.10 & 0.52 & & & & \\ -0.16 & 0.0001 & 0.005 & & & \\ 0.005 & -0.007 & 0.0003 & 0.0002 & & \\ 2.05 & -0.41 & 0.02 & 0.005 & 3.50 & \\ 0.18 & -0.05 & -0.008 & 0.00001 & 0.01 & 0.18 \end{pmatrix}$
$corr(\Omega_u)$	$\begin{pmatrix} 1.0 & & & & & \\ 0.02 & 1.0 & & & & \\ -0.32 & 0.002 & 1.0 & & & \\ 0.06 & -0.76 & 0.32 & 1.0 & & \\ 0.15 & -0.30 & 0.13 & 0.18 & 1.0 & \\ 0.06 & -0.17 & -0.26 & 0.01 & 0.01 & 1.0 \end{pmatrix}$

Note: Age shift and variance function intercept terms not highly correlated with polynomial terms. Polynomial terms not highly correlated justifies the unconstrained model.

Further developments

- More random classifications – higher levels and cross classifications
- Additional covariates with random coefficients in variance function
- Allow timing to depend on covariates (e.g. gender or social class) rather than incorporate these in the mean function.
  Thus e.g: $v_j = v_0 + T\alpha + w_j$
- Serial correlation between level 1 residuals can be allowed (Browne, W. and Goldstein, H. (2010). *Journal of Educational and Behavioral statistics* **35**: 453-473)
  Thus: $corr(e_{it_1}, e_{it_2}) = g(|t_1 - t_2|) \neq 0$
- Can be extended to multivariate and multiprocess models with additional responses, of possibly different types, especially at the individual level– e.g. adult measurements.